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Two-Parameter Skin-Friction Formula for Adiabatic Compressible Flow

Y. T. Chew*

Cambridge University, Cambridge, England

Introduction

THE derivation of a skin-friction formula depends on the similarity of velocity profiles. If the velocity profiles are similar from the wall to the edge of the boundary layer, a one-parameter skin-friction formula, expressing the relation between skin-friction coefficient C_f and Reynolds number Re, will be sufficient, e.g., those developed by Prandtl, Schlichting, von Karman, Schoenherr, etc. A one-parameter skin-friction formula is adequate to describe flat-plate boundary-layer flow where velocity profiles do collapse. However, for flows with pressure gradient or history effect, this is no longer valid. If the similarity breaks down everywhere within the boundary layer, then a correlated skin friction can be obtained by Clauser's[1] plot for the inner region of the velocity profile. If a more general formula which involves Reynolds number in term of freestream velocity is needed, some fairing of velocity profile to account for the nonsimilarity in the outer region must be included. The fairing can be done by using Coles'[2] wake function or Sarnecki's[3,4] intermittency function which give the C_f indirectly, or by Ludwieg and Tillmann's[5] method which yields an explicit formula for C_f. The fairing of velocity profile introduces yet another variable besides Re, i.e. wake parameter Π in Coles' method or shape factor H in Ludwieg and Tillmann's method. Hence, a two-parameter skin-friction formula is needed for flows with pressure gradient or history effect.

For compressible flow, the influence of Mach number has to be accounted for. Several authors (Winter and Gaudet,[6] Spalding and Chi,[7] Sommer and Short[8]) have attempted to include the Mach number effect as compressibility factors for C_f and Re_θ so that a one-parameter skin-friction formula for flat-plate flow can be derived empirically as in incompressible flow. In this paper, a two-parameter skin-friction formula for compressible adiabatic flow with pressure gradient or history effect will be developed.

Experiment

The experiment was carried out in a 114×165 mm blowdown-type wind tunnel at a Mach number of 2.5 to study the relaxing boundary layer after it had been disturbed by an oblique incident shock and Prandtl-Meyer expansion.[9] The adiabatic, turbulent boundary layer of about 6-mm thickness on an expansion-corner plate which had a sharp 6-deg turning angle was impinged by a plane oblique shock with flow deflection angle (α) of 4, 6, or 8 deg. The streamwise shock impingement position x_{sh} measured from the expansion corner can be varied from -60 mm to 60 mm. The boundary-layer Mach number profiles were measured by Pitot tube and the velocity profiles were derived by assuming a quadratic temperature relation. The skin-friction distribution was measured by Preston tube and Allen's[10] calibration curve was used for data reduction.

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*Graduate Student. Presently Visiting Assistant Professor, Southern Methodist University, Dallas, Texas.

Formulation

As mentioned before, the derivation of a skin-friction formula depends on wall similarity. The mixing-length theory and logarithmic law of the wall are mutually exclusive for flows with pressure gradient or history effect. The mixing-length theory[11] predicts no wall similarity. However, for incompressible flow near separation, Simpson et al.[12] have shown that the law of the wall still exists. The present data for compressible flow just downstream of a separation region also support the law of the wall when plotted in Van Driest's[13] transformed velocity, (Fig. 1), U^* where

$$U^* = \int_0^U \left(\frac{\rho}{\rho_w}\right)^{1/2} \mathrm{d}U$$

In this figure, the velocity profiles of this relaxing boundary layer at five x locations measured from the expansion corner are plotted on staggered ordinates. The dash lines are Coles' wake function with Π chosen to fit the data at the edge of the boundary layer. In view of the presence of a law of the wall, a skin-friction formula can be developed.

The two-parameter skin-friction formula was developed along similar lines to that of Ludwieg and Tillmann's[5] incompressible formula. This is done by approximating the transformed inner velocity profile by a power law as

$$\frac{U^*}{U_\tau} = f\left(\frac{U_\tau y}{\nu_w}\right) \tag{1}$$

which at $y=\theta^*$, where θ^* is the momentum thickness based on transformed velocity defined as

$$\theta^* = \int_0^\infty \frac{U^*}{U_e^*}\left(1-\frac{U^*}{U_e^*}\right)\mathrm{d}y$$

becomes

$$\frac{U_\tau}{U^*_{\theta^*}} = h\left(\frac{U^*_{\theta^*}\theta^*}{\nu_w}\right) \tag{2}$$

Equation (2) is only valid if the wall similarity applies up to $y=\theta^*$. Thus, a profile parameter γ which is equal to the value or projected value of $U^*_{\theta^*}/U_e$ at $y=\theta^*$ when Eq. (2) is valid, has to be introduced to account for the departure. By introducing

$$\gamma=\frac{U^*_{\theta^*}}{U_e},\quad U_\tau=\sqrt{\frac{\tau_w}{\rho_w}}=U_e\left(\frac{\rho_e}{\rho_w}\right)^{1/2}\sqrt{\frac{C_f}{2}},\quad Re_{\theta^*}=\frac{U_e\theta^*}{\nu_w}$$

Eq. (2) becomes

$$C_f=\gamma^2(\rho_w/\rho_e)G(Re_{\theta^*}\cdot\gamma) \tag{3}$$

where $G=2h^2$.

The function G can be derived by considering adiabatic compressible flows without pressure gradient. In this case, C_f is a function of Re_{θ^*} and M, and Eq. (3) becomes

$$F(Re_{\theta^*},M)=\gamma_0^2(\rho_w/\rho_e)_0 G(Re_{\theta^*}\cdot\gamma_0) \tag{4}$$

where subscript 0 denotes values at $\mathrm{d}P/\mathrm{d}x=0$. γ_0 depends on the shape of the velocity profile and is therefore a weak

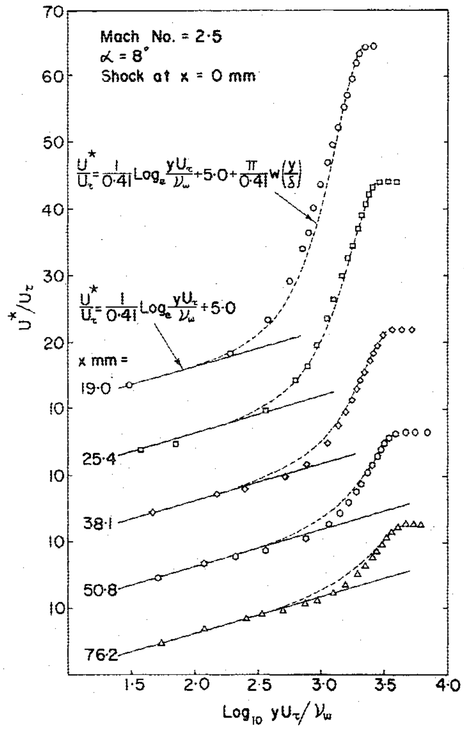

Fig. 1 A logarithmic law of the wall plot based on Van Driest's[13] transformed velocity.

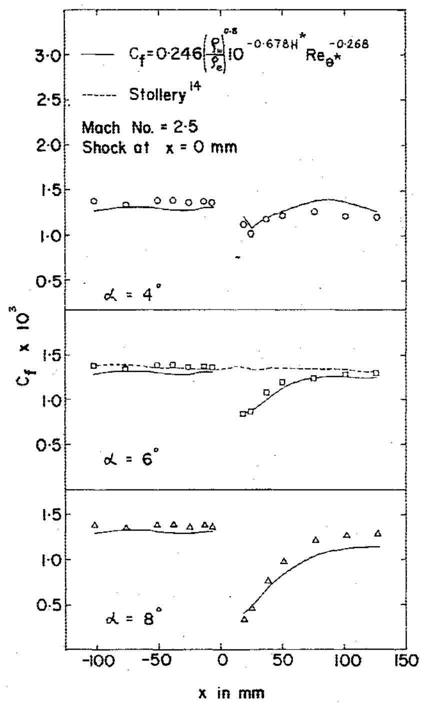

Fig. 2 Skin-friction coefficient C_f plot.

function of Reynolds number, i.e., $\gamma_0\ (Re_{\theta^*})$. Equations (3) and (4) can be shown to give[9]

$$C_f = \frac{\gamma^2}{\gamma_0^2(Re_{\theta^*})} F\left[Re_{\theta^*}\frac{\gamma}{\gamma_0(Re_{\theta^*})},\ M\right] \tag{5}$$

The function γ_0 and F can be obtained from flat-plate experimental data. As an approximation, one can assume that the conventional 1/7 power law is valid for the transformed velocity profile. Owing to the similarity of velocity profiles assumed, γ_0 will be independent of Reynolds number. The incompressible value of 0.717 was adopted since it agrees reasonably with the present flat-plate data. The corresponding compressible flat-plate skin-friction formula from the 1/7 power law is

$$C_f = F(Re_{\theta^*},\ M) = 0.0251\ Re_{\theta^*}^{\ -1/4}(\rho_w/\rho_e)^m \tag{6}$$

where the compressibility factor for C_f similar in form to that of Winter and Gaudet[6] is introduced. This equation was tested against the present flat-plate data and $m=0.8$ gave the best collapse of data. When $\gamma_0 = 0.717$ and Eq. (6) with $m=0.8$ are substituted into Eq. (5), it becomes

$$C_f = 0.0449(\rho_w/\rho_e)^{0.8}\gamma^{7/4}Re_{\theta^*}^{\ -1/4} \tag{7}$$

γ can be related empirically[5,9] to H^* by

$$\gamma = 2.333\times 10^{-0.398H^*} \tag{8}$$

where H^* is the shape factor based on transformed velocity defined by

$$H^* = \frac{\int_0^\infty \left(1-\frac{U^*}{U_e^*}\right)\mathrm{d}y}{\theta^*}$$

Substituting Eq. (8) into Eq. (7) gives

$$C_f = 0.246(\rho_w/\rho_e)^{0.8}10^{-0.678H^*}Re_{\theta^*}^{\ -0.268} \tag{9}$$

Discussion and Conclusion

Equation (9) is compared with data in Fig. 2 for shocks of different strength located at the expansion corner. It can be seen that it compares well with the experimental data calculated from Allen's[10] calibration curve, if one bears in mind that the calibration curve is obtained from floating element data which scatter by as much as ±12%. Unlike the one-parameter skin-friction formula,[14] it predicts the trend of C_f correctly in highly nonequilibrium flows.

It should be noted that the empirical constants proposed by Ludwieg and Tillmann[5] are retained here because there are not sufficient C_f data in the present investigation to justify any correlation. Besides, the data are secondary in nature, being deduced from a calibration curve. However, the various assumptions made by Ludwieg and Tillmann are verified at various stages of developing Eq. (9). This equation will collapse into the incompressible form as expected for compressible flow based on transformation theory.

The present derived extension of the Ludwieg-Tillmann formula to adiabatic compressible flow suffers from the same limitation as the original formula, namely it, in itself, is not a prediction formula. Predictions require use of the Karman integral to find Re_{θ^*} and some supplementary differential equation for the growth of H^*. Although there are many differential methods capable of predicting C_f for flows with pressure gradient, they usually fail to predict accurately downstream of abrupt changes, e.g., shock-wave and boundary-layer interaction. This was found to be so when the present data were compared[9] with Verma's[15] eddy-viscosity method and the Bradshaw et al.[16] turbulent shear method. Verma's method solves the enthalpy equation in addition to the momentum and continuity equations while Bradshaw's method assumes a quadratic enthalpy profile. Thus, the present formula provides useful means of estimating the local skin friction when the velocity profile has large history effects and where it is difficult to probe the boundary layer very close to the wall. However, it should be cautioned that a two-parameter formula implies a local equilibrium and, therefore not all history effects can be accounted for.

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Numerical Solutions of the Compressible Hodograph Equation

S. K. Liu* and W. L. Chow†
University of Illinois at Urbana-Champaign, Urbana, Ill.

SOLUTION of the hodograph equation has not been extensively explored for engineering purpose, even though this equation is linear, since the final physical configurations corresponding to these indirect solutions are often not of practical interest. However, a few configurations which assume shapes of straight-line profiles even in the two-dimensional geometry do have practical importance as a propulsive or metering device. Figure 1 shows one of such typical configurations. Early solutions of these problems for incompressible flow were obtained from free streamline theory with conformal mapping by von Mises.[1] The corresponding problem of compressible flow issuing from an orifice was solved by Busemann[2] by introducing the tangent gas approximation. In the interest of considering conical convergent nozzles as the primary propulsive device of an ejector system, the corresponding isoclines (line of constant flow angle) in the vicinity of the throat were approximated by Brown[3] and Brown and Chow[4] as obtained from the corresponding two-dimensional solution, which was again derived from conformal mapping after the tangent gas relationship was introduced. The sonic line for such a nozzle flow subsequently was established by calculations from the method of characteristics. These studies also clarified the choked or unchoked flows, depending upon whether the back pressure would influence the establishment of the sonic line. This method of analysis has been employed by Anderson[5,6] to evaluate the performance of aircraft ejector propulsive systems. Excellent agreement between the theoretical results and the experimental data clearly indicates that the approach is adequate for practical applications.

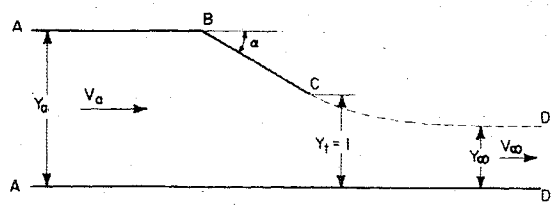

Fig. 1 Typical primary nozzle of an ejector system.

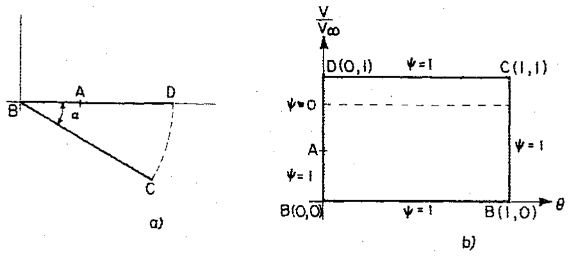

Fig. 2 The hodograph.

It is the intention of this Note to show that the compressible hodograph equation given by[7]

$$V^2\psi_{VV}+V(1+M^2)\psi_V+(1-M^2)/\alpha^2\psi_{\theta\theta}=0 \tag{1}$$

can be solved by numerical calculations for this type of problem. Variables ψ, V, and θ in Eq. (1) are already normalized by the respective reference quantities so that their range of variation is from zero to one.

It may be easily seen from Fig. 2 that the boundary values of ψ for the reactangular domain of V and θ are completely specified. The correct values of ψ within the domain may be established by the well-known successive over-relaxation scheme. Calculations may be terminated when the variation of ψ is less than an arbitrarily small value (e.g., 1.0×10^{-6}). Once this is established, the partial derivative of ψ_V and ψ_θ may be evaluated for all points including that on the boundary and the solution is interpreted back to the physical plane through integrating the following system of differential

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*Graduate Research Assistant, Department of Mechanical and Industrial Engineering. Present address, Battelle Laboratory, Columbus, Ohio. Student Member AIAA.

†Professor of Mechanical Engineering, Department of Mechanical and Industrial Enegineering. Associate Fellow AIAA.